Chapter 2
Quantum Mathematics

Summary The Vienna Experiments show that quantum indeterminacy has mathematical origins, rather than originating in any unknown physics. This chapter sets the tone for mathematical discipline needed in exposing those origins; and demonstrates an example of easy-to-access logic, visibly manifest in Quantum Mathematics, which textbook theory ignores.

2.1 Mathematical Discipline

In this book I often refer to the term *Quantum Mathematics*. By this, I mean the mathematics used in representing Quantum Theory. I make the distinction between Mathematics and Theory for the reason that explanations proposed in this book, which account for phenomena surrounding indeterminacy, concern Quantum Mathematics, rather than some missing part of Quantum Theory. Maybe surprisingly, the book's focus is on the *freedoms* that Quantum Mathematics permits, rather than any constraints it imposes.

Answers explaining quantum indeterminacy have long been there, hidden within Quantum Mathematics; but for the reason that Quantum Mechanics is done in the traditions of Applied Mathematics, those answers have remained obscured. However, the experimental work done in Vienna has forced the issue. The Vienna Experiments were evaluated from the standpoint of *Mathematical Logic*, by employing formalism known as a *formal system* to represent them.

When Quantum Mathematics is treated as a formal system, the machinery of indeterminacy becomes apparent. Exposing that machinery involves tracing through the derivation of Quantum Theory's formulae, to reveal *logically independent* information they contain. Briefly stated, logical independence refers to the logical disconnect that exists between items of mathematical information, which neither prove nor disprove one another.

This logical independence must be recognised formally, if Quantum Mathematics is to be made *truly* isomorphic with experiments. That means; in the usual way of doing Mathematical Physics, account must be kept of all *Postulates* and information they *assert*, *imply* or *prove* — the theory's *theorems.* But in addition, account must be kept, also, of information with which those *theorems* are merely *consistent* — that is to say: all information the *theorems* do not contradict, which also, they do not imply.

A vital point to realise is there is no contradiction in a theory consisting of information whose source is its axiom-set, plus extra information whose source is logically independent. Quantum Mathematics, in totality, will include axioms and their theorems — plus extra consistent formulae, deriving from logically independent machinery such as simultaneity, self-referential circularity or symmetry, not envisaged by Axioms.

The Vienna Experiments concern the measurement of photon polarisation. They demonstrate that the different behaviours of *predictable* outcomes of *pure states*, and *random* outcomes of *mixed states*[1], stem respectfully from an identifiable *logical connect*, and *logical disconnect*, conveyed in polarisation information.

It so happens that in Quantum Mathematics, there is corresponding *connect* and *disconnect*, which concurs with that empirical evidence. A good place for the reader to acquaint herself with the ideas is with a demonstration illustrating the mathematical difference between pure and mixed states, as it appears in the *free particle.*

2.2 Semantic Interpretation

In the quantum system known as the *free particle* the mathematics manifestly exhibits a logical difference, that distinguishes between pure and mixed states.

To see this, it is instructive to understand the difference between *syntactic information* versus *semantic information.* Syntax concerns rules obeyed in the manipulation or transformation of symbols and sentences[2]; where step-by-step, through a finite number of operations, in the manner of a machine algorithm, one formula is proved or derived from others. We then call this formula a *Theorem.* Examples are the proofs given in section 8.3.2.

Semantics, on the other hand, concern *interpretations*[3]. Here, interpretation does not refer to *physical* meaning, but to *mathematical* meaning. An

[1] A mixed state is any state, prior to measurement, which has an unpredictable outcome at measurement. Pure states are those that are perfectly predictable.

[2] A sentence is a certain kind of formula. See chapter 8 for an explanation.

[3] In Mathematical Logic, an *interpretation* is often known as a *model.*

interpretation is a mathematical structure consistent with axioms. Structures consistent with the Field Axioms are the fields of scalars: the rational field $\mathbb{Q}$, the real field $\mathbb{R}$, and the complex field $\mathbb{C}$. These are optional alternative interpretations. See table 8.1, page 52 for the Field Axioms.

Historically, in the traditions of Applied Mathematics, it has been customary to confer interpretation upon certain variables, in order to make them agree structurally with physical entities. Classically, position is designated a real quantity; and in Quantum Mechanics probability amplitude is complex. Questions I pose are: Are these interpretations demanded by the mathematics itself, or is there a freedom of choice? Is mathematical self-consistency reliant on the interpretation; or is the interpretation in the mind of the mathematician, only?

2.3 Possible Versus Necessary Information

Relevant to the machinery of indeterminacy, there are transition lines where one region of Quantum Mathematics, freely open to interpretation, interfaces with one that is rigidly not. That interface constitutes a 'step' separating *possible* interpretation from *necessary* interpretation.

This possible|necessary step is illustrated in the Quantum Mathematics of the quantum free particle. In order to reveal this step, position space and momentum space representations are considered to be one single, intrinsically linked, complementary system — accounting for the *whole information* of the free particle system. Then, by comparing the semantic information in the *pure states,* set against that in the *mixed states,* the *possible* versus *necessary* interpretations become apparent.

Consider the *pure state* eigenformulae:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\Phi\left(\mathsf{k}\right)\exp\left(+\mathrm{i}\mathsf{k}x\right)\right]=+\mathrm{i}\mathsf{k}\left[\Phi\left(\mathsf{k}\right)\exp\left(+\mathrm{i}\mathsf{k}x\right)\right] \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}k}\left[\Psi\left(\mathsf{x}\right)\exp\left(-\mathrm{i}k\mathsf{x}\right)\right]=-\mathrm{i}\mathsf{x}\left[\Psi\left(\mathsf{x}\right)\exp\left(-\mathrm{i}k\mathsf{x}\right)\right] \tag{2.2}$$

These two formulae are true, irrespective of any interpretation placed on the variable 'i'. Indeed, in these, 'i' can be interpreted as an arbitrary scalar. But in contrast, the *mixed state* pair of superpositions:

$$\Psi\left(x\right)=\int_{-\infty}^{+\infty}\left[\Phi\left(\mathsf{k}\right)\exp\left(+\mathrm{i}\mathsf{k}x\right)\right]\mathrm{d}\mathsf{k} \tag{2.3}$$

$$\Phi\left(k\right)=\int_{-\infty}^{+\infty}\left[\Psi\left(\mathsf{x}\right)\exp\left(-\mathrm{i}k\mathsf{x}\right)\right]\mathrm{d}\mathsf{x} \tag{2.4}$$

is true, only if we interpret ‘i’ as *pure imaginary.*

In the case of the pure state eigenformulae (2.1)–(2.2) the imaginary interpretation is purely in the mind of the theorist; but for the mixed state pair (2.3)–(2.4) the imaginary interpretation is implied in the mathematics. Whilst for the mixed state pair specific interpretation is *necessary*; for the pure state eigenformulae interpretation is *possible* but *not necessary.*

Remark 1. In writing (2.1)–(2.4), the italicised variables k, x are arbitrary scalars, and the san-serif notated k and x are rational or real (bound) dummy variables over the integrals. But there are a couple of issues of incidental interest, which must be said. In each of the pure state formulae (2.1)–(2.2), no validity is lost if k and x are permitted as arbitrary scalars. Furthermore, in the pure state formulae (2.1)–(2.2), the functions Φ and Ψ can be anything at all; in the mixed state pair (2.3)–(2.4) these must be members of the Hilbert space L^2.

2.4 Possible Versus Necessary Unitarity

Whether or not the variable ‘i’ is to be interpreted as the imaginary unit, determines whether the mathematics is unitary. The fact is that quantum theory for pure states need not be unitary or self-adjoint; whereas, for mixed states, unitarity is necessary. The step from pure states to mixed states represents a logical jump from *possible unitarity* to *necessary unitarity.*

Historically, the distinction between possible and necessary unitarity has not been noticed as any point of significance. No doubt, standard quantum theory ignores the fact with the intention of maintaining ‘consistent values’ across the theory. Following the traditions of Applied Mathematics, as if to iron out ‘the inconsistency’, an interpretation is applied across the whole of Quantum Theory, as an overall blanket condition. But, from the standpoint of any *formal system,* that designated interpretation is an *extraneous axiom.* What’s more, that axiom will constitute redundant information that obscures the true information that the mathematics genuinely asserts and conveys.

Rewriting (2.1)–(2.4) as formulae in *first order logic* shows there is no ‘value inconsistency’, and that the possible|necessary information can be conveyed by a single theory. Thus, for pure states:

$$\forall\eta \mid \ \frac{\mathrm{d}}{\mathrm{d}x}\left[\Phi\left(\mathsf{k}\right)\exp\left(\eta^{+1}\mathsf{k}x\right)\right] \ = \ \eta^{+1}\mathsf{k}\left[\Phi\left(\mathsf{k}\right)\exp\left(\eta^{+1}\mathsf{k}x\right)\right] \tag{2.5}$$

$$\forall\eta \mid \ \frac{\mathrm{d}}{\mathrm{d}k}\left[\Psi\left(\mathsf{x}\right)\exp\left(\eta^{-1}\mathsf{x}k\right)\right] \ = \ \eta^{-1}\mathsf{x}\left[\Psi\left(\mathsf{x}\right)\exp\left(\eta^{-1}\mathsf{x}k\right)\right] \tag{2.6}$$

And for mixed:

$$\exists\eta \mid \ \Psi(x) = \int_{-\infty}^{+\infty} \left[\Phi(\mathsf{k}) \exp\left(\eta^{+1}\mathsf{k}x\right)\right] \mathrm{d}\mathsf{k} \tag{2.7}$$

$$\exists\eta \mid \ \Phi(k) = \int_{-\infty}^{+\infty} \left[\Psi(\mathsf{x}) \exp\left(\eta^{-1}\mathsf{x}k\right)\right] \mathrm{d}\mathsf{x} \tag{2.8}$$

These formulae are consistent because there is no contradiction between the universal quantifier $\forall\eta$ in (2.5)–(2.6) and the existential quantifier $\exists\eta$ in (2.7)–(2.8).

2.5 Breaking Free of the Unitary Postulate

Whilst (2.5)–(2.8) depict a mathematical system that is self-consistent, which truthfully and faithfully represents both pure and mixed states; that faithfulness is had at the expense of quantum theory's most treasured Principle. These new formulae violate the *Quantum Postulate* imposing unitary|Hermicity; because the formulae (2.5)–(2.6), and *Postulate,* are in contradiction. Specifically, (2.5)–(2.6) are not unitary $\forall\eta$, but only for the two values: $\eta = \pm\mathrm{i}$.

Not to worry. The postulated unitary|Hermicity is not needed. Unitarity is implied where it is needed: in the mathematics of mixed states, implicit in complementarity. Elsewhere unitary|Hermicity is redundant.

Once free of the *Unitary Postulate*, the imaginary unit no longer exists axiomatically across the whole theory, but only where implied by (2.7)–(2.8).

Finally, the step-transition from pure states in (2.5)–(2.6) to mixed states in (2.7)–(2.8) is picked up in Chapter 13; and as we shall see there, once its axiomatic existence is discarded, a *logically independent* imaginary unit is exposed. Likewise in Chapter 14, we shall see analogous behaviour in the Pauli system.

2.6 Principles for the Conveyance of Whole Information

In Mathematical Logic, *'necessary information* versus *possible information'* is recognised as constituting a Modal Logic. Yet, in textbook Quantum Theory, the distinction separating possible from necessary is not especially noticeable, nor is it recognised; and the logical distinction separating pure states from mixed states is lost.

In order to reveal logical information missed by standard Quantum Theory, I propose strict observance of the following Principles:

- Utterly disregard all physical meaning.

† Treat formulae purely as statements in mathematics.

- Treat complementary systems as a single system.

† In textbook Wave Mechanics it is acceptable to perform calculations in either complementary system: either in momentum space or in position space, say. That approach yields qualitative and quantitative information but misses logical information. Complementary systems coexist with information flowing between them. They must be considered as a single system.

- Observe what is *actually* asserted by the mathematics.

† The mathematician may not introduce new information, not derivable from formulae already in place, without recording the fact in an account of assumptions made. For instance, assignment of scalars to the real line, say, potentially brings in new information from outside the mathematics.

- Be wary of making inadvertent assumptions which the algebra cannot express.

† When physicists set out a problem they begin by agreeing a convention of right-handed reference frames for 3–space. Yet, the algebra knows nothing of this right-handedness, nor even can it represent it.

2.7 Avoiding Confusion over Terminology

2.7.1 Orthogonal

Before going further, it is important to clear up any confusion over my use of the term **'orthogonal'**. Risk of confusion arises through the fact there are certain vector spaces, notably the Lie algebras[4], whose basis vectors are *matrices*, and whose span covers all possible linear combinations of those matrices. And so the term 'orthogonal' might ambiguously apply to pairs of matrices as basis vectors, or, to matrices themselves, as being members of the Orthogonal Group.

Unless I state otherwise, I am referring to orthogonality between vectors. More specifically, my use of the term refers to a zero dot product; in this form:

$$\mathsf{e}_i\mathsf{e}_j + \mathsf{e}_j\mathsf{e}_i = \mathbb{0} \tag{2.9}$$

[4] Lie algebras are vector spaces equipped with commutator products between basis vectors. A well-known example is the algebra of the Canonical Commutation Relation.

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fundamental. Whilst generally I use the terms unitary and unitary|Hermitian interchangeably, fundamentally, I am talking about *unitarity*.

2.7.3 Fields and Scalars

Elementary Algebra is the bedrock algebra upon which Quantum Mathematics rests; it is the algebra of *fields* and *scalars*. This Algebra and its fields and scalars play a huge part in the story of indeterminacy.

The fields and scalars of Elementary Algebra are not to be confused with those conceptualised in Physics; although in some circumstances they may actually be the same things. In Physics, scalars are first rank tensors (whose transformations are invariant) and fields are arrangements of objects spread throughout some space or other. The fields and scalars of Elementary Algebra are objects that satisfy the Axioms of Elementary Algebra — the *Field Axioms*.