Chapter 4
Logical Independence

Summary This chapter introduces ideas of logical independence.

4.1 The 3–valued Logic of Hans Reichenbach

The story of logical independence in Quantum Theory might be regarded as having begun in 1944, when the problem of indeterminacy was confronted by Hans Reichenbach [23], and whose ideas were subsequently supported by Hilary Putnam [22]. Reichenbach constructed a purely invented logic, based on the logic of Łukasiewicz, which now can be seen to agree with Boolean mathematics discovered by Tomasz Paterek et al.

Reichenbach's book details a '3–valued logic' comprising values: true, false and *indeterminate;* possessing the feature: 'true' is not the same as 'not false'. He showed that this non-classical logic resolves 'causal anomalies' of quantum theory, including *complementarity,* and the *action at a distance* paradox, highlighted by Einstein, Podolsky & Rosen [7, 11].

Fig. 4.1: Hans Reichenbach resolved *'causal anomalies'* in quantum experiments using his *'3–valued logic'*, where *true* is not the same as *not false.*

Reichenbach is not in opposition to the 'mainstream' quantum logics, based on Postulates of Hilbert space theory, such as Birkhoff and von Neumann [5]. The approach of Reichenbach was to design a logic, isomorphic to the epistemology for *prepared* and *measured* states – typically the question of what we may know about the state of a photon immediately before measurement. As argued by Hardegree, Reichenbach's logic is framework for an alternative formulation of Quantum

Theory [9]. Reichenbach was predicting, or at least anticipating, something of the nature of the Vienna Team's findings.

4.2 What is Logical Independence?

In Mathematical Logic, a *formal system* is a system of mathematical formulae, treated as propositions, where focus of interest is on *provability* and *non-provability*. Within such a formal system, any two propositions are **either:** *logically dependent* — in which case, one proves or disproves the other; or each proves or disproves the other — **or otherwise:** they are *logically independent*, in which case, neither proves nor disproves the other.

A helpful perspective on this is the viewpoint of Gregory Chaitin's information-theoretic formulation [6]. In that, logical independence is seen in terms of information content. If a proposition contains information, not contained in some given set of axioms, then those axioms can neither prove nor disprove the proposition.

Edward Russell Stabler explains logical independence in the following terms. A formal system is a postulate-theorem structure; the term postulate being synonymous with axiom. In this structure, there is discrimination, separating *assumed* from *provable* statements. Any statement labelled as a postulate which is capable of being proved from other postulates should be relabelled as a theorem. And if retained as a postulate, it is logically superfluous and redundant [30]. If incapable of being proved or disproved from other postulates, it is logically independent.

Gödel's term: *Mathematical undecidability* is identical in meaning to logical independence under certain circumstances. Undecidability refers solely to independence present in axiom systems which cannot be completed, Elementary Algebra under the Field Axioms, for instance; as such, Gödel's First Incompleteness Theorem applies. Independence present in axiom systems which are completable, cannot be said to a consequence of Gödel.

4.3 Axiom Systems

Given some mathematical language or other, that of the Paterek et al Boolean propositions, or, Elementary Algebra, say; we can imagine a set that contains all that language's statements. Various of those statements will *prove* others. And certainly there will be statements that disprove others too; because, for every statement there is another which is its negation, and all statements disprove their negations. Those remaining statements, none of which prove or

Page not included

of them. Any definite handedness would need to be specified from outside the system — an *ingression* of information

4.6 Ingression[1], Self-reference and Accident

In section 4.3, I said an axiom is an independent statement of the language, whose negation has been contradicted, by the fact it was necessary in the derivation of some formula, which is adopted as true. In this way, the formula is a composite of all the necessary independent statements that make it up. Those statements, we might take the liberty of calling Axioms.

Then, with that formula in tact, if a further additional independent statement is *needed* in deriving a yet further formulae, of yet greater complexity, I call the admission of that extra independent information an *'ingress'* or *'ingression'*. Such an ingress is needed in section 2.3, in passing from pure to mixed states.

As and when we derive formulae, we have to be careful because sometimes, ingress of newly introduced independent information is not so noticeable and easy to miss. It may not be clear that new information has ingressed at all, from somewhere other than Axioms.

Combinations of Axioms are used to prove *theorems* — and our understanding of theorems is that they are formulae containing information, originating only from Axioms. Yet, starting from Axioms alone, and no other statements, there are certain *extraordinary* constructs which 'prove' formulae, which are *apparently* theorems, but which in fact, contain extra information, logically independent of Axioms.

This kind of logical independence stems from the arrangement by which Axioms are threaded together in procuring a 'proof'. For instance, the introduction of simultaneous equations — in the guise of matrix transformations — can introduce independence, as exemplified in section 7.3. Another type involves self-referential circularity, as exemplified in chapters 13 and 14.

These kinds of simultaneity and circularity constitute an important class of logical independence whose *ingression* involves *accident of coincidence*. That is to say, circumstances which are not guaranteed and occur by chance of coincidence.

[1] Ingression applies to information; nothing physical enters.