Chapter 7
Logical Independence in Applied Mathematics

Summary This chapter begins the story of how logically independent theory exists within Quantum Mathematics.

7.1 Logical Independence is Everywhere

The theorems of Kurt Gödel point the way forward, toward revealing logical independence in Quantum Mathematics — which agrees with the 3–valued logic proposed by Reichenbach.

It so happens that wherever there are scalars axiomatised under the Axioms of Elementary Algebra — the *Field Axioms* — a 3–valued logic is exerted by this Algebra, which is the same as the logic of Reichenbach, but whose criterion is not *truth*, but *provability*.

Naturally, unless suppressed in some way, that logic will be inherited by Applied Mathematics and must express itself in Quantum Mathematics. And indeed, currently, that logic is suppressed by the Quantum Postulate enforcing the unitary|Hermitian environment. That suppression is an issue addressed comprehensively in subsequent chapters.

In the Vienna Experiments, the system of Boolean propositions used, can be *completed*. Those propositions, I introduce in section 4.4 and cover the use of in chapter 6. The fact they can be completed means there is a finite limit to the number of logically independent statements belonging to that system. For that reason, that Boolean system is not subject to Kurt Gödel's Incompleteness Theorems.

In contrast, the *Field Axioms* can never be completed; they prescribe a *first order theory*, and as such, Gödel's First Incompleteness Theorem (1931) guarantees a limitless number of statements, all logically independent of the Field Axioms. Consequently, Elementary Algebra contains a limitless number of statements, all logically independent. Conveniently, Gödel's Completeness Theorem (1929), together with the Soundness Theorem, provide a test confirming, whether any given statement in the language is logically independent of the Field Axioms.

Gödel referred to his logically independent statements as: *mathematically undecidable*. But some care is needed in use of these two terms. In arithmetics to which Gödel's First Incompleteness Theorem applies, independence and undecidability describe the same concept. However this cannot be said for systems of propositions to which Gödel's First Incompleteness Theorem does not apply.

Elementary Algebra, Linear Algebra, Applied Mathematics, Mathematical Physics and Quantum Mathematics are all first-order theories. They are all consequent on the Field Axioms; and consequently, they all inherit the logically independent statements affected by the Field Axioms. There is nothing exotic in this; it is easy to find independent statements in any of these theories. Simple examples can be seen in section 8.3.

7.2 The Logically Independent Role of the Imaginary Unit

Asking the question: 'By what mathematical mechanism does the imaginary unit enter Quantum Mathematics, and, what is its root origin?' leads to Big Answers explaining quantum indeterminacy.

In addition to the Field Axioms, another certain statement is insisted upon and adopted by Standard Quantum Theory, as an extra axiom. That extra axiom asserts the *a priori* existence of the imaginary unit; and thus, lays down the field of complex scalars as ontology. The claim of Standard Theory is that probability amplitudes are *a priori* ontologically complex.

Deviating entirely from that official approach, if the imaginary unit axiom is deleted and not allowed, the field of complex scalars is free to exist *a posteriori*[1]; by implication, as consequence of any logically independent mechanism that demands it. The deletion of that extra axiom, it turns out, is critically important in representing indeterminacy, because, a logically independent imaginary unit plays a vital role in representing indeterminate information in Quantum Mathematics.

[1] An *a priori* fact is one assumed to be a fundamental truth, for the sake of developing an argument. An *a posteriori* fact is provable from facts already established.

Checking that the imaginary unit is indeed logically independent is done using a test provided by Gödel's Completeness Theorem (1929), together with its converse, the Soundness Theorem. The detail of that is covered in chapter 8.

7.3 Machinery Necessitating the Imaginary Unit

The imaginary unit is a *fixed-point,* resulting as consequence of certain mathematical machines, acting on arrays of scalars. The machines are transformations and the arrays of scalars are vectors. Typically these machines are self-referential mappings, logically independent of the *Field Axioms;* which means they are able to operate spontaneously, since nothing obstructs their operation.

There are two 'Mechanisms' by which a logically independent imaginary unit may emerge in Elementary Algebra; both arise out of Linear Algebra.

In ***Mechanism One*** certain eigenvalue equations are responsible. These are particular *orthogonal rotations.* Somewhat paradoxically they seek to map vectors *parallel* to themselves! For example[2]:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} \mathsf{x} \\ \mathsf{y} \end{bmatrix} \mapsto \lambda \begin{bmatrix} \mathsf{x} \\ \mathsf{y} \end{bmatrix} \tag{7.1}$$

where the numbers and variables in this formula are all scalars, consistent with the Field Axioms. Written in the language of Elementary Algebra, (7.1) becomes the simultaneous pair of linear equations:

$$\mathsf{y} = -\lambda \mathsf{x} \tag{7.2}$$

$$\mathsf{y} = \lambda^{-1} \mathsf{x} \tag{7.3}$$

This pair of linear equations (7.2)–(7.3) can be true only if:

$$\exists \lambda \left(\lambda^2 = -1 \right) \tag{7.4}$$

But, no such λ exists as logical consequence of the Field Axioms, because the square root of minus one does not follow as consequence from them. Indeed, there is no theorem of the Field Axioms which asserts existence of this number; but equally, no theorem contradicts its existence either; and so (7.4) is not implied by the Field Axioms, and nor is it denied; and therefore, it is logically

[2] Written as a self-referential mapping (7.1) takes the form of a machine mapping a vector to itself:

$$\left[\frac{1}{\lambda} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \begin{bmatrix} \mathsf{x} \\ \mathsf{y} \end{bmatrix} \mapsto \begin{bmatrix} \mathsf{x} \\ \mathsf{y} \end{bmatrix}$$

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invariant under any similarity transformation; and hence, the trace of E is equal to the trace of $\mathsf{W}^{-1}\mathsf{E}\mathsf{W}$. And secondly: from the fact that these traces are the negatives of one another, as shown thus:

$$
\begin{aligned}
& & \mathsf{WE} + \mathsf{EW} &= \mathbb{0} \\
&\Rightarrow & \mathsf{WE} &= -\mathsf{EW} \\
&\Rightarrow & \mathsf{E} &= -\mathsf{W}^{-1}\mathsf{EW} \\
&\Rightarrow & \mathrm{tr}\,(\mathsf{E}) &= \mathrm{tr}\left(-\mathsf{W}^{-1}\mathsf{EW}\right) \\
&\Rightarrow & \mathrm{tr}\,(\mathsf{E}) &= -\mathrm{tr}\left(\mathsf{W}^{-1}\mathsf{EW}\right)
\end{aligned}
$$

Combining those two facts, we deduce the trace of E is zero; and likewise, by similar proofs, so too are the traces of W and A. This narrows down the type of matrix that may represent anticommuting operators. That said; the converse is not true; not all trace zero operators are anticommuting. And so, if A, E and W are written as general trace zero, 2×2 matrices, thus:

$$
\mathsf{A} = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & -\mathfrak{a} \end{pmatrix} \qquad \mathsf{E} = \begin{pmatrix} \mathfrak{e} & \mathfrak{f} \\ \mathfrak{g} & -\mathfrak{e} \end{pmatrix} \qquad \mathsf{W} = \begin{pmatrix} \mathfrak{w} & \mathfrak{x} \\ \mathfrak{y} & -\mathfrak{w} \end{pmatrix}
$$

then particular values for these will be anticommuting matrices, all mutually orthogonal with one another.

My line of attack now is to demonstrate that: if matrices A, E and W are to obey all three of the anticommutation relations in (7.5), then a component $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ somewhere in that matrix system is unavoidable.

Begin by writing each of A, E and W expanded as a linear combination of independent matrices; in the case of A, the most obvious of which is:

$$
\mathsf{A} = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & -\mathfrak{a} \end{pmatrix} = \mathfrak{a}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \mathfrak{b}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathfrak{c}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
$$

Yet there is a more appropriate expansion

$$
\mathsf{A} = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & -\mathfrak{a} \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
$$

where

$$
a = \mathfrak{a} \qquad b = (\mathfrak{c} + \mathfrak{b})/2 \qquad c = (\mathfrak{c} - \mathfrak{b})/2
$$

and where there are exactly similar expansions for E and W.

The strategy now is to make the assumption that Proposition 1 is false; that (7.5) is fully satisfied by particular values of A, E and W whose $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ components are zero; and then deduce conditionality for that assumption. So, assume:

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