Chapter 12
Redundancy of the Unitary Postulate

Summary Chapter 8 shows the imaginary unit is a logically independent scalar. Insight into the mechanisms of indeterminacy reduces to understanding how and where Quantum Mathematics drives the necessity for this number's presence in the theory. This chapter eliminates fundamental symmetry as a source; and shows that (complex) unitarity arises logically independently, due to complementarity.

12.1 The Status of Unitarity in Quantum Theory

Historically, the reason given, requiring that quantum theories be unitary, is the universal need for preserved invariance of probability amplitudes. Authors making explicit mention of this rule are Sakurai [27, p401] and Robinett [24, p277]. Interpretationally; from this universality we infer that fundamental symmetries in Nature are ontologically unitary. The expectation is that all symmetries at the Foundations of Quantum Mechanics are unitary, and if any symmetry is not unitary, it is not a fundamental symmetry of Quantum Mechanics. This would indicate that unitarity should be regarded as a blanket condition, covering all group theories representing quantum systems, along with the whole of Quantum Mathematics following from them; and that this unitarity should be asserted formally as an *Axiomatic Postulate* — *a priori*. In practice, this is generally assured by requiring Hermitian observables.

However, in contradiction to that *a priori* view of unitarity, findings of this chapter show that, for one fundamental symmetry at least, unitarity is an *a posteriori* consequence of complementarity. The distinction is important because the *a posteriori* unitarity has logical consequences for Quantum Mathematics and Quantum Theory, which the *a priori* unitarity does not.

Below I examine a fundamental symmetry at the Foundations of Wave Mechanics — the homogeneity of space — and show that unitarity is *logically* *independent* of homogeneity; it being implied elsewhere by complementarity. This means that preserved probability amplitudes (and realness of observable eigenvalues) are guaranteed by complementarity, without the need for any Axiomatic Postulate imposing unitarity|Hermicity. This single counter-example removes the reasons for imposing unitarity|Hermicity — *by Postulate* — and renders it *redundant.*

12.2 Homogeneity of Space and Wave Mechanics

Textbook theory says that the *Canonical Commutation Relation* derives from the homogeneity of space. This chapter shows the Canonical Commutation Relation does not derive from homogeneity, but derives from a duality viewpoint of homogeneity, seen both from the viewpoints of position space and of momentum space, combined. Additionally, a specific particular fixed scale factor, relating position space with momentum space is necessary. That scaling is plus or minus the imaginary unit. It is this additional scaling information which enables *self-consistent* complementarity between the system variables which makes the system unitary. Without this particular scaling, the Canonical Commutation Relation is left non-unitary and broken.

The Canonical Commutation Relation:

$$\mathbf{px} - \mathbf{xp} = -i\hbar$$

embodies core algebra at the heart of Wave Mechanics. With general acceptance amongst quantum theorists, the professed significance of this relation is that it derives from the *homogeneity of space* — and is *unitary.* Here, I now re-examine the Canonical Relation's derivation and establish that the homogeneity symmetry is of itself *not* unitary. And in consequence establish that the Canonical Commutation Relation does not, itself, faithfully represent homogeneity, but contains extra unitary information also.

Imposing homogeneity on a system is identical to imposing a null physical or geometrical effect, under arbitrary translation of reference frame. To formulate this arbitrary translation, resulting in null effect, the principle we invoke is *Form Invariance.* This is the concept from relativity that symmetry transformations leave formulae fixed in *form*, though *values* may alter [26]. In the case at hand, the relevant formula whose form is held fixed is the eigenvalue equation for position. The Dirac notation is not used, because Hilbert space is not assumed:

$$\mathbf{x} f_{\mathsf{x}}(x) = \mathsf{x} f_{\mathsf{x}}(x)\,. \tag{12.1}$$

$$
\begin{tikzcd}
\mathbf{x} f_{\mathsf{x}} = \mathsf{x} f_{\mathsf{x}} \arrow[r, "\text{translation}", "O_{\mathsf{x}} \to O_{\mathsf{x}'}"'] \arrow[d] & \mathbf{x} f_{\mathsf{x}+\epsilon} = (\mathsf{x}+\epsilon)\, f_{\mathsf{x}+\epsilon} \arrow[d, "f_{x+\epsilon} \to \psi_{x+\epsilon} \to \psi_{\mathsf{x}}"', "\text{similarity}"] \\
\left[\mathsf{S}\mathbf{x}\mathsf{S}^{-1} - \epsilon\mathbb{1}\right] \psi_{\mathsf{x}} = \mathbf{x}\psi_{\mathsf{x}} & \left[\mathsf{S}\mathbf{x}\mathsf{S}^{-1} - \epsilon\mathbb{1}\right] \psi_{\mathsf{x}} = \mathsf{x}\psi_{\mathsf{x}} \arrow[l]
\end{tikzcd}
$$

Fig. 12.1: Scheme of transformations. The bottom left hand formula is the resulting group relation.

In (12.1), the san-serif x denotes the eigenvalue, along with the label on the eigenvector f_{x}; the variable x (curly) is the function domain. The use of two different variables here, may seem unusual and pointless. In fact, logically they are different; x is quantified *existentially* but x is quantified *universally.* The overall scheme of transformations is depicted in Figure 12.1.

With the form of (12.1) held fixed as the reference system is displaced, variation in the position operator $\mathbf{x}$ determines a group relation, representing the homogeneity symmetry. Under arbitrarily small displacements, this group corresponds to the linear algebra representing homogeneity locally. These are homogeneity's Lie group and Lie algebra.

To maintain the form of (12.1) under translation, the basis $\{f_{\mathsf{x}}\}$ is cleverly managed: whilst the translation transforms the basis from $\{f_{\mathsf{x}}\}$ to $\{f_{\mathsf{x}+\epsilon}\}$, a similarity transformation is also applied, chosen to revert $\{f_{\mathsf{x}+\epsilon}\}$ back to $\{f_{\mathsf{x}}\}$. In this way f_{x} is held static. Actually, similarity transforms can be found only for the class of functions: $\left\{\psi_{\mathsf{x}} \in L^1(\mathbb{R})\right\} \subset \{f_{\mathsf{x}}\}$. These are the functions in Banach space — complete normed spaces with no inner product. Hilbert space, which is a subspace of the Banach space, is not needed at this point.

The similarity transformation to be used is a one-parameter subgroup of the general linear group: $\mathsf{S}(\epsilon) \subset \mathsf{S} \in \mathsf{GL}(\mathbb{F})$, where parameter ϵ identically coincides with the displacement parameter, and $\mathbb{F}$ is any infinite field. The only restrictions on $\mathsf{S}(\epsilon)$, therefore, are that S be invertible and ϵ be a scalar.

In textbook theory, our understanding is that we must *also* insist on $\mathsf{S}(\epsilon)$ being made unitary. Thought to be *intrinsically necessary*, unitarity is imposed axiomatically — *by Postulate.* It is this point, where the Canonical Commutation Relation, as we know it, finds its unitary origins. However, this imposed unitarity is additional information, extra to the information of homogeneity. In consequence, the underlying symmetry beneath the whole of Wave Mechanics is not homogeneity of space, but instead, a unitary subsymmetry of it.

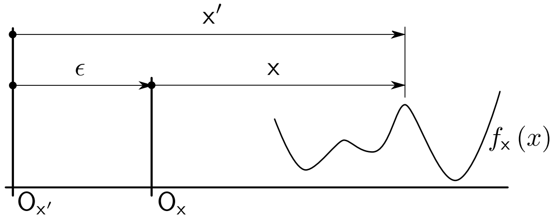

Fig. 12.2: **Passive translation of a function** Two reference systems, O_{x} and $\mathsf{O}_{\mathsf{x}'}$, arbitrarily displaced by ϵ, individually act as reference systems for position of a function f_{x}. If the x-space is homogeneous, then regardless of the value of ϵ, physics concerning this function is described by formulae whose form remains invariant, though values may change. **Note:** The function and reference frames are not epistemic; f_{x} is non-observable and O_{x} and $\mathsf{O}_{\mathsf{x}'}$ are not observers.

In what follows, it shall become clear that homogeneity is not unitary; that the Canonical Commutation Relation is unitary for other reasons; and that the axiomatic unitarity imposed here is redundant.

I now proceed as if doing an experiment whose aim is to discover the point where the mathematics unavoidably becomes unitary. The method is to allow $\mathsf{S}(\epsilon)$ it's widest generality, so that the *whole information* of homogeneity is faithfully conveyed through the mathematics.

The experiment begins with the position eigenvalue equation (12.1) being rewritten, in the form of a quantified proposition (12.2). Dirac notation is not in use, to avoid any inference that vectors are intended as orthogonal, in Hilbert space, or equipped with a scalar product; none of these is implied.

Consider the eigenformulae for position operator $\mathbf{x}$, eigenfunctions f_{x} and eigenvalues x, seen from the reference frame O_{x}:

$$\forall x \exists \mathbf{x} \exists \mathsf{x} \exists f_{\mathsf{x}} \mid \quad \mathbf{x} f_{\mathsf{x}}(x) = \mathsf{x} f_{\mathsf{x}}(x) \tag{12.2}$$

Homogeneity demands existence of an equally relevant frame $\mathsf{O}_{\mathsf{x}'}$ translated through an arbitrary displacement ϵ. See Figure 12.2. *Form Invariance* then guarantees a formula for $\mathsf{O}_{\mathsf{x}'}$ of the same form as that for O_{x} in (12.2), thus:

$$\forall x' \exists \mathbf{x}' \exists \mathsf{x}' \exists f'_{\mathsf{x}'} \mid \quad \mathbf{x}' f'_{\mathsf{x}'}(x') = \mathsf{x}' f'_{\mathsf{x}'}(x') \tag{12.3}$$

The first point to be said about (12.2) and (12.3) is that the operator $\mathbf{x}$ is an invariant on the homogeneity symmetry, and does not depend on the eigenvalue or measurement outcome. Hence $\mathbf{x}' = \mathbf{x}$, and (12.3) becomes:

$$\forall x' \exists \mathbf{x} \exists \mathsf{x}' \exists f'_{\mathsf{x}'} \mid \quad \mathbf{x} f'_{\mathsf{x}'}(x') = \mathsf{x}' f'_{\mathsf{x}'}(x') \tag{12.4}$$

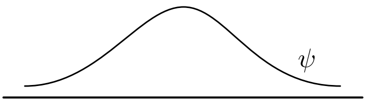

Fig. 12.3: The linear transformations S exist only for bounded ψ_{x}, maximally, the Banach space $L^1(\mathbb{R})$. These are the Lebesgue integrable functions: $\int_{\mathsf{x}} |\psi_{\mathsf{x}}|$ is finite.

Translation: Applying the translation first. This transforms position, thus:

$$\forall \mathsf{x}' \exists \mathsf{x} \mid \quad \mathsf{x} \mapsto \mathsf{x}' = \mathsf{x} + \epsilon \tag{12.5}$$

and transforms the function, thus:

$$\forall x \forall f'_{\mathsf{x}'} \exists f_{\mathsf{x}+\epsilon} \exists x' \mid \quad f_{\mathsf{x}}(x) \mapsto f'_{\mathsf{x}'}(x') = f_{\mathsf{x}+\epsilon}(x - \epsilon) \tag{12.6}$$

Substituting (12.5) and (12.6) into (12.4) gives the translated formula:

$$\forall x \exists \mathbf{x} \exists \mathsf{x} \exists f_{\mathsf{x}+\epsilon} \mid \quad \mathbf{x} f_{\mathsf{x}+\epsilon}(x - \epsilon) = (\mathsf{x} + \epsilon) f_{\mathsf{x}+\epsilon}(x - \epsilon). \tag{12.7}$$

Similarity: Now applying the similarity transformation. This involves the one-parameter linear operator $\mathsf{S}(\epsilon)$. Any such transformation would be invalid if it were to result in an unbounded f_{x}. Valid transformations $\mathsf{S}(\epsilon)$ exist only if there exists a function space $\{\psi_{\mathsf{x}}\}$, which is complete, normalisable, not restricted to separable[1] functions; and of course, it should be a subset of the translatable functions $\{f_{\mathsf{x}}\}$. Such function spaces are well-known; they are the normed L^1 spaces, known as Banach spaces. See Figure 12.3. Hilbert space L^2 is a particular class of Banach space whose norm is determined by an *inner product*. Homogeneity does not demand any inner product; so an L^1 Banach space is sufficient.

The following transformation is valid for all operators $\mathsf{S}(\epsilon)$ that map Banach spaces $\{\psi_{\mathsf{x}}\}$ into themselves.

$$\forall x \forall \psi_{\mathsf{x}+\epsilon} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \mathsf{S}(\epsilon)^{-1} \psi_{\mathsf{x}}(x) = \psi_{\mathsf{x}+\epsilon}(x - \epsilon). \tag{12.8}$$

Remark 13. Banach spaces and transformation operators acting on them can be entirely real, demonstrating the mathematics is not essentially unitary at this point. Nonetheless, in standard quantum theory, $\mathsf{S}(\epsilon)$ would be unitary, set by the mathematician; and doing so would restrict the space of functions ψ_{x} to the Hilbert space L^2 without homogeneity demanding it.

Now acting on the translation (12.7) with (12.8) to form the similarity transformation, noting the restriction to the space $\{\psi_{\mathsf{x}}\}$.

$$\forall x \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \mathsf{S}(\epsilon)\, \mathbf{x} \mathsf{S}(\epsilon)^{-1} \psi_{\mathsf{x}}(x) \ = (\mathsf{x} + \epsilon)\, \psi_{\mathsf{x}}(x)$$

[1] Separable means countable, as are the integers, as opposed to continuous, like the reals.

Introducing the trivial eigenformulae: $\forall\psi_{\mathsf{x}}\forall x\forall\epsilon \mid \epsilon\mathbb{1}\psi_{\mathsf{x}}(x) = \epsilon\psi_{\mathsf{x}}(x)$ and subtracting:

$$\forall x\exists\mathbf{x}\exists\mathsf{x}\exists\psi_{\mathsf{x}}\exists\mathsf{S} \mid \quad \left[\mathsf{S}(\epsilon)\,\mathbf{x}\mathsf{S}(\epsilon)^{-1} - \epsilon\mathbb{1}\right]\psi_{\mathsf{x}}(x) = \mathsf{x}\psi_{\mathsf{x}}(x)\,. \tag{12.9}$$

Comparing the original position eigenformula (12.2) against the transformed one (12.9), we deduce the group relation for homogeneity:

$$\forall x\exists\mathbf{x}\exists\psi_{\mathsf{x}}\exists\mathsf{S} \mid \quad \mathbf{x}\psi_{\mathsf{x}}(x) = \left[\mathsf{S}(\epsilon)\,\mathbf{x}\mathsf{S}(\epsilon)^{-1} - \epsilon\mathbb{1}\right]\psi_{\mathsf{x}}(x)\,. \tag{12.10}$$

From this group relation, the commutator for the Lie algebra is now computed. Because $\mathsf{S}(\epsilon)$ is a one-parameter subgroup of $\mathsf{GL}(\mathbb{F})$, there exists a unique linear operator $\mathbf{g}$ for real parameters ϵ, such that:

$$\forall\mathsf{S}\exists\mathbf{g} \mid \quad \mathsf{S}(\epsilon) = \mathrm{e}^{\epsilon\mathbf{g}} \tag{12.11}$$

Noting that homogeneity is totally independent of scale, an arbitrary scale factor η is extractable, thus: $\forall\mathbf{g}\forall\eta\exists\mathbf{k} : \mathbf{g} = \eta\mathbf{k}$, implying:

$$\forall\eta\forall\mathsf{S}\exists\mathbf{k} \mid \quad \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\eta\epsilon\mathbf{k}} \tag{12.12}$$

$$\forall\eta\forall\mathsf{S}\exists\mathbf{k} \mid \quad \mathsf{S}^{-1}_{(\epsilon)} = \mathsf{S}_{(-\epsilon)} = \mathrm{e}^{-\eta\epsilon\mathbf{k}} \tag{12.13}$$

Remark 14. The test for whether this theory is unitary is to check whether the exponentials in (12.12) and (12.13) are reciprocals of one another.

Substitution of (12.12) and (12.13) into (12.10) gives:

$$\begin{aligned}
&\forall x\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \mid && \exp(+\eta\epsilon\mathbf{k})\,\mathbf{x}\exp(-\eta\epsilon\mathbf{k})\,\psi_{\mathsf{x}}(x) = [\mathbf{x}+\epsilon\mathbb{1}]\,\psi_{\mathsf{x}}(x)\\
\Rightarrow\ &\forall x\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \mid && \left[\mathbb{1}+\eta\epsilon\mathbf{k}+\mathcal{O}\left(\epsilon^2\right)\right]\mathbf{x}\left[\mathbb{1}-\eta\epsilon\mathbf{k}+\mathcal{O}\left(\epsilon^2\right)\right]\psi_{\mathsf{x}}(x) = [\mathbf{x}+\epsilon\mathbb{1}]\,\psi_{\mathsf{x}}(x)\\
\Rightarrow\ &\forall x\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \mid && \left[\mathbf{x}+\eta\epsilon\mathbf{k}\mathbf{x}+\mathcal{O}\left(\epsilon^2\right)\right]\left[\mathbb{1}-\eta\epsilon\mathbf{k}+\mathcal{O}\left(\epsilon^2\right)\right]\psi_{\mathsf{x}}(x) = [\mathbf{x}+\epsilon\mathbb{1}]\,\psi_{\mathsf{x}}(x)\\
\Rightarrow\ &\forall x\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \mid && \left[\mathbf{x}+\eta\epsilon\mathbf{k}\mathbf{x}-\eta\epsilon\mathbf{x}\mathbf{k}+\mathcal{O}\left(\epsilon^2\right)\right]\psi_{\mathsf{x}}(x) = [\mathbf{x}+\epsilon\mathbb{1}]\,\psi_{\mathsf{x}}(x)\\
\Rightarrow\ &\forall x\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \mid && [\mathbf{k}\mathbf{x}-\mathbf{x}\mathbf{k}]\,\psi_{\mathsf{x}}(x) = \left[\eta^{-1}\mathbb{1}-\mathcal{O}(\epsilon)\right]\psi_{\mathsf{x}}(x)
\end{aligned}$$

At the limit, as $\epsilon \to 0$, we have:

$$\forall x\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \mid \quad [\mathbf{k},\mathbf{x}]\,\psi_{\mathsf{x}}(x) = \eta^{-1}\mathbb{1}\psi_{\mathsf{x}}(x) \tag{12.14}$$

And by an analogous proof, similar to all that above, but conditional upon the existence of eigenfunctions $\chi_{\mathsf{k}}(k)$ of $\mathbf{k}$:

$$\forall k\forall\zeta\exists\chi_{\mathsf{k}}\exists\mathbf{x}\exists\mathbf{k} \mid \quad [\mathbf{x},\mathbf{k}]\,\chi_{\mathsf{k}}(k) = \zeta^{-1}\mathbb{1}\chi_{\mathsf{k}}(k)\,. \tag{12.15}$$

Individually, each of the formulae (12.14) and (12.15) are separate consequences of the homogeneity symmetry, and yet they are not the Canonical Commutation Relation. Importantly, there is no assurance they offer complementarity. That is to say, there is no guarantee they can both be simultaneously valid; or that they are mutually consistent.

12.3 The New Independent Unitary Information

If homogeneity is to imply the Canonical Commutation Relation, new information is needed, in addition to (12.14) and (12.15). For one thing, quantifiers $\forall\eta$ in (12.14) and $\forall\zeta$ in (12.15) contradict the Canonical Commutation Relation. Hence, some extra condition that restricts these is necessary information. It should be emphasised that this extra condition will be new information that is *logically independent* of homogeneity.

I proceed by making the assumption that the extra information needed, is for both these formulae to be valid — simultaneously. As they appear, there is no guarantee of that. Note that (12.14) is quantified $\exists\psi_{\mathsf{x}}$, and (12.15) quantified $\exists\chi_{\mathsf{k}}$. And so their combined quantification is $\exists\psi_{\mathsf{x}}\exists\chi_{\mathsf{k}}$; it is not $\forall\psi_{\mathsf{x}}\exists\chi_{\mathsf{k}}$ or $\forall\chi_{\mathsf{k}}\exists\psi_{\mathsf{x}}$. Hence, non-contradictory values for ψ_{x} and χ_{k} are not guaranteed; any happy coincidence between them would be accidental.

In precise terms, to uncover the extra information that guarantees simultaneity, I pose the assumed simultaneity formally as an hypothesis, then proceed to deduce conditionality implied by it. Essentially, the hypothesis is an experiment needing guesswork, and it seems likely that, vectors ψ_{x} and χ_{k} must be particular parallel scalings of one another.

Hypothesised coincidence:

$$\forall\chi_{\mathsf{k}}\forall\zeta\forall\eta\exists\psi_{\mathsf{x}} \;\wedge\; \forall x\exists k \mid \quad \chi_{\mathsf{k}}(k) = \zeta\eta\psi_{\mathsf{x}}(x) \tag{12.16}$$

Taking (12.14) and the negative of (12.15) gives us the pair:

$$\forall x\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \mid \quad [\mathbf{k},\mathbf{x}]\,\psi_{\mathsf{x}}(x) = +\eta^{-1}\mathbb{1}\psi_{\mathsf{x}}(x) \tag{12.17}$$

$$\forall k\forall\zeta\exists\chi_{\mathsf{k}}\exists\mathbf{x}\exists\mathbf{k} \mid \quad [\mathbf{k},\mathbf{x}]\,\chi_{\mathsf{k}}(k) = -\zeta^{-1}\mathbb{1}\chi_{\mathsf{k}}(k) \tag{12.18}$$

Substituting the **Hypothesised coincidence** (12.16) into (12.18) gives the pair:

$$\forall x\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \mid \qquad [\mathbf{k},\mathbf{x}]\,\psi_{\mathsf{x}}(x) = +\eta^{-1}\mathbb{1}\psi_{\mathsf{x}}(x) \tag{12.19}$$

$$\forall x\forall\zeta\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \mid \quad \zeta\eta\,[\mathbf{k},\mathbf{x}]\,\psi_{\mathsf{x}}(x) = -\eta^{+1}\mathbb{1}\psi_{\mathsf{x}}(x) \tag{12.20}$$

Subtracting (12.19) and (12.20):

$$\forall x\forall\zeta\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \mid \quad \left\{(\zeta\eta-1)\,[\mathbf{k},\mathbf{x}] + \left(\eta+\eta^{-1}\right)\mathbb{1}\right\}\psi_{\mathsf{x}}(x) = \mathbb{0} \tag{12.21}$$

The formula (12.21) is **self-contradictory**, because it cannot be true for all values of ζ and η. In truth, (12.21) is valid only for values:

$$\zeta = \pm \mathrm{i} \qquad \eta = \mp \mathrm{i} \tag{12.22}$$

This confirms there is something invalid about the **Hypothesis** (12.16). Nonetheless, an **Adjusted Hypothesis** (12.23), in which quantifiers $\forall\zeta\forall\eta$ are replaced by $\exists\zeta\exists\eta$, thus:

$$\forall \chi_{\mathsf{k}} \exists \zeta \exists \eta \exists \psi_{\mathsf{x}} \ \wedge \ \forall x \exists k \mid \quad \chi_{\mathsf{k}}(k) = \zeta \eta \psi_{\mathsf{x}}(x) \tag{12.23}$$

eliminates the self-contradiction, thus:

$$\forall x \exists \zeta \exists \eta \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid \quad \left\{ (\zeta\eta - 1)[\mathbf{k}, \mathbf{x}] + \left(\eta + \eta^{-1}\right) \mathbb{1} \right\} \psi_{\mathsf{x}}(x) = \mathbb{0} \tag{12.24}$$

Summarising

On top of homogeneity, logically independent, extra new information is needed in constructing the Canonical Commutation Relation:

$$[\mathbf{k}, \mathbf{x}] = -i\mathbb{1} \quad \text{or} \quad [\mathbf{p}, \mathbf{x}] = -i\hbar\mathbb{1} \tag{12.25}$$

That information is represented in the steps taken in going from the non-unitary (12.14) and (12.15) to the unitary (12.25). Precisely, the Canonical Commutation Relation does not represent the homogeneity of space; it represents homogeneity for a particular scaling between position space and wavenumber space (momentum space).

12.4 Conclusion

The above establishes that the homogeneity of space, or indeed, the homogeneity symmetry itself is not the source of unitary information in Wave Mechanics. That is to say, the foundational symmetry we suppose to be the fundamental ontology of this quantum system unitary. Rather, unitarity is separate, logically independent of the underlying ontology, and a condition implied within complementarity.

And therefore, if the reason given for postulating that quantum theory should be unitary or self-adjoint, is that symmetries in Nature are intrinsically, unavoidably and ontologically unitary, then this one counter-example requires that a different reason be found, or otherwise, the *Postulate* be withdrawn.

This does not mean Quantum Theories are not unitary, because certainly they are; it means that unitarity may not be imposed by the mathematician, for the reason she believes unitarity to be a Fundamental Physical Principle.