Chapter 8
Verifying Logical Independence of the Imaginary Unit

Summary This chapter explains and demonstrates logical independence of the imaginary unit, in relation to the Field Axioms.

8.1 Elementary Algebra as Formal System

Elementary Algebra is a *first order theory.* Typically, first order theories describe properties and objects on domains. They incorporate constants, variables, terms and functions. They are very familiar to engineers and physicists.

Here, I treat Elementary Algebra as a *formal system* or *formal theory*[1]. My reason for doing so is that the full logic is conveyed and expressed explicitly.

A *formal system* comprises: a *formal language*, rules for writing formulae (propositions) and further rules of deduction. Information is designated in two levels of compulsion. *Propositions* assert information that is questionable. And *Axioms* are propositions presupposed to be 'true'; they are postulates adopted *a priori.*

Vital is the use of quantifiers: *there-exists* ($\exists$) and *for-all* ($\forall$). Quantifiers convey important information, missed out in ordinary equations; their employment eliminates unintended ambiguities relating to domain. For instance: the equation $y = x^2$ doesn't express whether $\forall y \exists x \left(y = x^2\right)$, or $\forall x \exists y \left(y = x^2\right)$, is intended. Yet, logically, these two are very different. In addition to quantifiers, formal systems also employ the *logical connectives: not* ($\neg$), *and* ($\wedge$), or ($\vee$), *implies* ($\Rightarrow$) and *if-and-only-if* ($\Leftrightarrow$); and also the turnstile symbols: *derives* ($\vdash$) and *models* ($\models$).

[1] Good references for physicists on formal theories are: Edward Stabler's book, *An introduction to mathematical thought* [30]; the book by Wei Li, *Mathematical Logic* [17]. For a course on the subject: The Open University, *M381 Mathematical Logic* [21].

8.2 The Field Axioms

Elementary algebra is the abstraction of the familiar arithmetic used to combine numbers in the rational, real and complex number systems, through the operations of *addition* and *multiplication*. These number systems are the *infinite fields*. Rules for their algebra are the *Field Axioms*.

An efficient axiom-set is a selection of propositions, all logically independent of one another. Axioms chosen to do a particular job won't necessarily be unique, and there may be slight variation between those chosen by different authors. Table 8.1 lists the axiom-set chosen here. They are the well-known, conventional Field Axioms, appended with the additional axiom Inf, which guarantees zero is unique and the finite fields are excluded by denying modulo arithmetic. This ensures that Elementary Algebra covers only the *infinite fields*.

Remark 9. It is interesting to speculate whether these axioms might have some physical significance in Nature.

The Field Axioms

	Additive Group		
A0	$\forall\beta\forall\gamma\exists\alpha \mid$	$\alpha = \beta + \gamma$	Closure
A1	$\exists 0\forall\alpha \mid$	$\alpha + 0 = \alpha$	Identity 0
A2	$\forall\alpha\exists\beta \mid$	$\alpha + \beta = 0$	Inverse
A3	$\forall\alpha\forall\beta\forall\gamma \mid$	$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$	Associativity
A4	$\forall\alpha\forall\beta \mid$	$\alpha + \beta = \beta + \alpha$	Commutativity
	Multiplicative Group		
M0	$\forall\beta\forall\gamma\exists\alpha \mid$	$\alpha = \beta \times \gamma$	Closure
M1	$\exists 1\forall\alpha \mid$	$\alpha \times 1 = \alpha$	Identity 1
M2	$\forall\beta\exists\alpha \mid$	$\alpha \times \beta = 1 \wedge \beta \neq 0$	Inverse
M3	$\forall\alpha\forall\beta\forall\gamma \mid$	$(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$	Associativity
M4	$\forall\alpha\forall\beta \mid$	$\alpha \times \beta = \beta \times \alpha$	Commutativity
Am	$\forall\alpha\forall\beta\forall\gamma \mid$	$\alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma)$	Distributivity
Inf	$\forall\alpha \mid$	$\alpha \neq 0 \wedge \alpha + 1 \neq 0 \Rightarrow \alpha(\alpha + 1) \neq 0$	Unique zero

Table 8.1: The Field Axioms. These are written as sentences in *first-order logic*. The axiom Inf is included to exclude all finite fields, that is, all modulo arithmetic; meaning that the field of rationals is the smallest. Collectively, Axioms assert a definite set of information, deriving a definite set of theorems. Any proposition (in the language) is either a theorem or is otherwise logically independent. Theorems include *proofs* and *negations*; all other statements in the language are logically independent.

Collectively, the Field Axioms assert a definite set of information, by deriving a definite set of theorems. Any proposition (in the language) is either a theorem, or is otherwise logically independent of them. Note that proof of negation is a theorem. And so, any given formula (in the language) can be regarded as a proposition in Elementary Algebra, that may prove to be a theorem, derivable from the Field Axioms, or otherwise, without exception, be logically independent.

Which of these is actually the case can (in principle) be diagnosed in a process that compares information in the given proposition or formula, against information contained in the Field Axioms. This would involve attempting to derive that proposition from the Field Axioms, to discover that: **either** it *is* a theorem; **or otherwise**, extra information is needed to complete its derivation, that Field Axioms cannot provide — demonstrating logical independence. But as a *test* for logical independence, that process will generally be inconclusive. For a *definitive test* of logical independence, the Soundness & Completeness Theorems from Mathematical Logic must be used.

8.3 Examples of Logical Independence in Elementary Algebra

The propositions (8.1)–(8.5) are five examples illustrating the three distinct *provability* values possible, under the Field Axioms.

Notice that these formulae do not assert equality; they assert *existence*. Each is a proposition asserting existence for some instance of a variable α, complying with an equality, specifying a particular numerical value.

$$\exists\alpha \mid \alpha = 3 \tag{8.1}$$

$$\exists\alpha \mid \alpha^2 = 4 \tag{8.2}$$

$$\exists\alpha \mid \alpha^2 = 2 \tag{8.3}$$

$$\exists\alpha \mid \alpha^2 = -1 \tag{8.4}$$

$$\exists\alpha \mid \alpha^{-1} = 0 \tag{8.5}$$

Of the five examples, the Field Axioms prove (8.1) and (8.2). Proofs are given below. Also, the Field Axioms prove the negation of (8.5) because Axiom M2 contradicts (8.5). The remaining two, (8.3) and (8.4), are neither proved nor negated; they are logically independent of the Field Axioms. Demonstration of that independence is dealt with in sections 8.4–8.5.

Accordingly, the instance of α in (8.5) is *inconsistent* with the Field Axioms; the instances of α in (8.1)–(8.4) are all numbers whose existences are *consistent* with the Field Axioms; but in addition to being consistent, existences of α in (8.1) and (8.2) are also *proved* or *derived* by the Field Axioms.

And accordingly, the instance of α in (8.5) is rejected as *necessarily* non-existent; instances of α in (8.1) and (8.2) are accepted as scalars, proved to

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Example One

$$\begin{aligned} & \forall\beta\forall\underline{\gamma}\exists\alpha \mid \ \alpha = \beta + \gamma \\ & \forall\kappa\forall\lambda\underline{\exists\gamma} \mid \ \gamma = \kappa + \lambda \\ \Rightarrow\ & \forall\beta\forall\kappa\forall\lambda\exists\alpha \mid \ \alpha = \beta + \kappa + \lambda \end{aligned}$$

Example Two

$$\begin{aligned} & \forall\beta\underline{\forall\gamma}\exists\alpha \mid \ \alpha = \beta + \gamma \\ & \forall\beta\underline{\exists\gamma} \mid \ \gamma = 2\beta \\ \Rightarrow\ & \forall\beta\exists\alpha \mid \ \alpha = \beta + 2\beta \\ \Rightarrow\ & \forall\eta\exists\alpha \mid \ \alpha = \eta + 2\eta \qquad \beta \curvearrowright \eta \end{aligned}$$

8.3.2 Proofs

Proposition (8.1) is derivable from the Field Axioms.

Proof.

$\exists 1\forall\kappa \mid \kappa \times 1 = \kappa$	Axiom M1	
$\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta + \gamma$	Axiom A0	(8.6)
$\forall\beta\exists\alpha \mid \alpha = \beta + \beta$	$\gamma \curvearrowright \beta$ (8.6)	(8.7)
$\forall\beta\exists\gamma \mid \gamma = \beta + \beta$	$\alpha \curvearrowright \gamma$ (8.7)	(8.8)
$\forall\beta\exists\alpha \mid \alpha = \beta + \beta + \beta$	Substitute (8.8), (8.6)	(8.9)
$\exists\alpha \mid \alpha = 1 + 1 + 1$	Freely Assign β=1, (8.9)	(8.10)

Hence, $\exists\alpha \mid \alpha = 3$ is proved derivable. □

Proposition (8.2) is derivable from the Field Axioms.

Proof.

$\exists 1\forall\kappa \mid \kappa \times 1 = \kappa$	Axiom M1	(8.11)
$\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta + \gamma$	Axiom A0	(8.12)
$\forall\beta\exists\alpha \mid \alpha = \beta + \beta$	$\gamma \curvearrowright \beta$ (8.11)	(8.13)
$\forall\alpha \mid \alpha \times \alpha = \alpha \times \alpha$	identity rule	(8.14)
$\forall\beta\exists\alpha \mid \alpha \times \alpha = (\beta + \beta) \times (\beta + \beta)$	Substitute (8.13), (8.14)	(8.15)
$\exists\alpha \mid \alpha \times \alpha = (1 + 1) \times (1 + 1)$	Freely Assign β=1, (8.15)	(8.16)
$\exists\alpha \mid \alpha \times \alpha = ((1 + 1) \times 1) + ((1 + 1) \times 1)$	Axiom AM, (8.16)	(8.17)
$\exists\alpha \mid \alpha \times \alpha = (1 + 1) + (1 + 1)$	Axiom M1, (8.17)	(8.18)
$\exists\alpha \mid \alpha \times \alpha = 1 + 1 + 1 + 1$		(8.19)

Hence, $\exists\alpha \mid \alpha^2 = 4$ is proved derivable. □

8.4 Model Theory: Soundness & Completeness

Model theory is a branch of Mathematical Logic applying to all first-order theories; which includes Elementary Algebra. Our interest is in two standard theorems: the Soundness Theorem and its converse, the Completeness Theorem, as well as theorems that follow from them. These theorems formalise the link connecting truth or consistency (semantic information), with provability (syntactic information) between formulae. In combination these theorems identify an *excluded middle*, comprising the set of all non-provable, non-negatable propositions — those logically independent of Axioms.

Any first-order axiom-set is *modelled* by particular mathematical structures, consistent with it. These are *closed* structures, consistent with each and every axiom of that axiom-set. In the case of the Field Axioms used in this book (Table 8.1), those modelling structures are the *infinite fields.* Each field consists of numbers known as *scalars,* with all scalars complying with the rules of arithmetic, set out by the Field Axioms. A proposition's logical independence from Axioms is confirmed by demonstrating disagreement, between *any two* of the Axioms' models

Of relevance to Quantum Theory is proposition (8.4); this is true in the complex plane, but false in the real line.

Theorem 1. *The Soundness Theorem:*

$$\mathit{\Sigma} \vdash \mathcal{S} \Longrightarrow \forall \mathcal{M} \left(\mathcal{M} \models \Sigma \Rightarrow \mathcal{M} \models \mathcal{S} \right). \tag{8.20}$$

If $\mathcal{M}$ is any structure that models axiom-set $\mathit{\Sigma}$, and $\mathit{\Sigma}$ derives sentence $\mathcal{S}$, then every $\mathcal{M}$ also models $\mathcal{S}$.

Alternatively**:** *If a sentence is a theorem, provable under an axiom-set, then that sentence is true for every model of that axiom-set.*

Theorem 2. *The Completeness Theorem:*

$$\mathit{\Sigma} \vdash \mathcal{S} \Longleftarrow \forall \mathcal{M} \left(\mathcal{M} \models \Sigma \Rightarrow \mathcal{M} \models \mathcal{S} \right). \tag{8.21}$$

If $\mathcal{M}$ is any structure that models axiom-set $\mathit{\Sigma}$, and every $\mathcal{M}$ models sentence $\mathcal{S}$, then *$\mathit{\Sigma}$ derives $\mathcal{S}$.*

Alternatively**:** *If a sentence is true for every model of an axiom-set, then that sentence is a theorem, provable under that axiom-set.*

8.4.1 Logical Dependence

Jointly, Theorems 1 and 2 imply the 2-way implications, in Theorem 3:

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