Chapter 13
Logically Independent Unitarity in the Free Particle

Summary In Chapter 2, I used the *quantum free particle* to demonstrate how mixed states are unavoidably unitary, while pure states need not be. The job *this* chapter does is demonstrate how an *uncaused, accidental* self-referential mechanism facilitates a logical step-transition, from those non-unitary pure states to the unitary mixed states. And how, in this self-referential system, complementarity is an inherent consequence of the self-consistency which is necessary. Effectively, the procedure I work through is a formal logical approach to the derivation of the Fourier Integral Theorem.

13.1 Pure State Superpositions

I begin with eigenformulae for pure state superpositions. They are formulae taken from (2.5) and (2.6) which I re-write here, but with all quantifiers stated explicitly:

$$\forall\eta\forall x\forall k\forall\Phi \mid \eta^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\left[\Phi\left(k\right)\exp\left(\eta^{+1}kx\right)\right] = k\left[\Phi\left(k\right)\exp\left(\eta^{+1}kx\right)\right] \tag{13.1}$$

$$\forall\eta\forall x\forall k\forall\Psi \mid \eta^{+1}\frac{\mathrm{d}}{\mathrm{d}k}\left[\Psi\left(x\right)\exp\left(\eta^{-1}xk\right)\right] = x\left[\Psi\left(x\right)\exp\left(\eta^{-1}xk\right)\right] \tag{13.2}$$

My claim is that these two non-unitary eigenformulae *faithfully* represent pure state superpositions for momentum and position: (13.1) representing pure state, momentum superpositions in position space; and (13.2) representing pure state, position superpositions in momentum space. The crucial and manifest difference from standard theory is the arbitrary η that replaces the imaginary unit. You can see that these two formulae share certain variables. And so they are linked to that extent. Nevertheless, it is perfectly legal to consider these formulae, and they do work perfectly, as separate individuals which do not communicate with one another. That is to say, no complementarity is invoked or demanded.

But now, I intend to build the superpositions $\Psi(x)$ and $\Phi(k)$ when complementarity *is* forced upon position and momentum. Note that in (13.1) and (13.2), $\Psi(x)$ and $\Phi(k)$ can be any functions at all; there is no restriction.

Consider now the following pair of formulae[1].

$$\forall\eta\forall x\forall a\exists\Psi \mid \quad \Psi(x) = \textstyle\int_{\mathsf{k}} \left[\exp\left(\eta^{+1} x\mathsf{k}\right) a(\mathsf{k})\right] \;\wedge\; a \in L^1(\mathbb{R}) \tag{13.3}$$
$$\forall\eta\forall k\forall b\exists\Phi \mid \quad \Phi(k) = \textstyle\int_{\mathsf{x}} \left[\exp\left(\eta^{-1} k\mathsf{x}\right) b(\mathsf{x})\right] \;\wedge\; b \in L^1(\mathbb{R}) \tag{13.4}$$

In writing these, the san-serif notated k and x are the dummy (bound) variables over the integrals. The italicised variables η, k, x, a, b are all bound variables under the existential and universal quantifiers $\exists$ and $\forall$. I have laid out the ordering of variables to mirror the convention of repeated dummy indices used in summations of discrete quantities, familiar in matrix transformations. This is done to emphasise that the exponential integrals are integral linear operators acting on the amplitudes. Note that these formulae do not assert equality, they assert *existence*. Also note that the integrals exist, and the pair of propositions are guaranteed true if a and b are functions restricted to the Banach space $L^1(\mathbb{R})$.

I state without proof, that all four formulae (13.1)–(13.4) are derivable from Axioms of Elementary Algebra — the *Field Axioms* — listed in Table 8.1, entirely from provable steps, but with the caveat that the derivations never terminate, due to infinitely many steps being involved; as is the nature of irrational scalars. In this respect these derivations fall short of being proofs, although proof to any arbitrary accuracy is possible; as is the nature of rational scalars. The point here is that all information in (13.1)–(13.4), other than the fact of non-terminability, derives from information asserted in the Field Axioms.

13.2 Self-Reference in the Free Particle

I now explore the possibility of (13.3)–(13.4) accepting information, circularly, from one another, through a mechanism where $a(\mathsf{k})$ feeds off $\Phi(k)$ and $b(\mathsf{x})$ feeds off $\Psi(x)$. This is self-reference that circulates information around a *crossover-loop*; continually passing that information back and forth, between position space and momentum space. There is no *cause* implying this self-reference; the idea is that it is *prevented* by nothing. It's simply that the *Field Axioms* do not contradict this circularity, but are consistent with it. Indeed, the self-referential information is logically independent of all algebraic rules in operation — so is available through chance accident.

[1] I use the notation $\int_{\mathsf{k}} f(\mathsf{k}) = \int_{-\infty}^{+\infty} f(\mathsf{k})\,\mathsf{dk}$.

To proceed, the strategy followed will be to posit a hypothesis that such self-reference can occur, then investigate for conditionality implied. To properly document this assumption, the hypothesis is formally declared, thus:

Hypothesised coincidence:

$$\forall\Phi\exists a \mid \quad a = \Phi; \tag{13.5}$$

$$\forall\Psi\exists b \mid \quad b = \Psi. \tag{13.6}$$

Note: there is no guarantee that any such coincidence should exist; if so, it will be by chance accident; we proceed to investigate. When this assumed hypothesis is substituted[2] into (13.3)–(13.4) we get:

$$\forall\eta\forall x\forall\Phi\exists\Psi \mid \quad \Psi(x) = \textstyle\int_{\mathsf{k}} \left[\exp\left(\eta^{+1}x\mathsf{k}\right)\Phi(\mathsf{k})\right] \ \wedge \ \ \Phi \in L^1(\mathbb{R}) \tag{13.7}$$

$$\forall\eta\forall k\forall\Psi\exists\Phi \mid \quad \Phi(k) = \textstyle\int_{\mathsf{x}} \left[\exp\left(\eta^{-1}k\mathsf{x}\right)\Psi(\mathsf{x})\right] \ \wedge \ \ \Psi \in L^1(\mathbb{R}) \tag{13.8}$$

These facilitate *cross-substitution*[3] of Φ and Ψ. And that permits circularity in the form of a *crossover-loop,* a species of *multi-loop* self-reference, where functions Ψ and Φ are evaluated in terms of themselves:

$$\begin{array}{r|l} \forall\eta\forall x & \\ \forall\Psi\exists\Psi & \Psi(x) = \textstyle\int_{\mathsf{k}} \left[\exp\left(\eta^{+1}x\mathsf{k}\right)\int_{\mathsf{x}}\left[\exp\left(\eta^{-1}\mathsf{k}\mathsf{x}\right)\Psi(\mathsf{x})\right]\right] \ \wedge \ \ \Psi \in L^1(\mathbb{R}) \end{array} \tag{13.9}$$

$$\begin{array}{r|l} \forall\eta\forall k & \\ \forall\Phi\exists\Phi & \Phi(k) = \textstyle\int_{\mathsf{x}} \left[\exp\left(\eta^{-1}k\mathsf{x}\right)\int_{\mathsf{k}}\left[\exp\left(\eta^{+1}\mathsf{x}\mathsf{k}\right)\Phi(\mathsf{k})\right]\right] \ \wedge \ \ \Phi \in L^1(\mathbb{R}) \end{array} \tag{13.10}$$

The formulae (13.9)–(13.10) show an apparent conflict or schism in quantifiers: in the combinations $\forall\Psi\exists\Psi$ and in $\forall\Phi\exists\Phi$. This may be a weakness in the formalism which was never designed for self-referential situations. Actually, both $\forall\Psi$ and $\exists\Psi$ are correct, and likewise, $\forall\Phi$ and $\exists\Phi$; the $\forall$ and $\exists$ quantifiers apply separately to different instances of the same variable. To clarify, from here on, the vectors shall be written with distinct suffices thus: $\Psi_\forall$, $\Psi_\exists$ and $\Phi_\forall$, $\Phi_\exists$.

In (13.9)–(13.10), to the extent to which the integrals exist, the integral signs can be moved through to the left, with no effect on the arithmetic. Then also; by Tonelli's Theorem, reversing their order preserves their values. Writing them as mappings, these formulae tidy up to become:

$$\begin{array}{r|l} \forall\eta\forall x & \\ \forall\Psi\exists\Psi & \textstyle\int_{\mathsf{x}}\int_{\mathsf{k}} \exp\left[\left(\eta^{+1}x + \eta^{-1}\mathsf{x}\right)\mathsf{k}\right]\Psi_\forall(\mathsf{x}) \longmapsto \Psi_\exists(x) \ \wedge \ \ \Psi \in L^1(\mathbb{R}) \end{array} \tag{13.11}$$

$$\begin{array}{r|l} \forall\eta\forall k & \\ \forall\Phi\exists\Phi & \textstyle\int_{\mathsf{k}}\int_{\mathsf{x}} \exp\left[\left(\eta^{-1}k + \eta^{+1}\mathsf{k}\right)\mathsf{x}\right]\Phi_\forall(\mathsf{k}) \longmapsto \Phi_\exists(k) \ \wedge \ \ \Phi \in L^1(\mathbb{R}) \end{array} \tag{13.12}$$

□

[2] See section 8.3.1 for substitution involving quantifiers.

[3] Self-reference in these spaces forces orthogonality on the vectors. See chapter 16

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$$\Psi(x) = \int_{\mathsf{k}} \exp(+\mathrm{i}x\mathsf{k})\,\Phi(\mathsf{k}) \qquad \Phi(\mathsf{k}) = \int_{\mathsf{x}} \exp(-\mathrm{i}\mathsf{k}\mathsf{x})\,\Psi(\mathsf{x}) \tag{13.20}$$

Prior to making that substitution, the very first step is to insert the Gaussian into the left formula. Essentially, in the limiting case, this Gaussian is the identity function:

$$\mathbb{1}(\mathsf{k}) = \lim_{\epsilon\to 0}\left[\exp\left(-\frac{\epsilon}{2}\mathsf{k}^2\right)\right] \tag{13.21}$$

flattening out to unity over the entirety of $\mathsf{k} \in \mathbb{R}$ as $\epsilon \to 0$. The reason the Gaussian is chosen over k is that this is the variable over which convergence is in doubt.

Inserting the Gaussian into the integrand:

$$\begin{aligned}\Psi(x) = \int_{\mathsf{k}} \exp(+\mathrm{i}x\mathsf{k})\,\Phi(\mathsf{k}) &= \int_{\mathsf{k}} \mathbb{1}(\mathsf{k}) \exp(+\mathrm{i}x\mathsf{k})\,\Phi(\mathsf{k}) \\ &= \int_{\mathsf{k}} \left[\lim_{\epsilon\to 0} \exp\left(-\frac{\epsilon}{2}\mathsf{k}^2\right) \exp(+\mathrm{i}x\mathsf{k})\,\Phi(\mathsf{k})\right]\end{aligned}$$

And substituting for $\Phi(\mathsf{k})$:

$$\Psi(x) = \int_{\mathsf{k}} \left[\lim_{\epsilon\to 0} \exp\left(-\frac{\epsilon}{2}\mathsf{k}^2\right) \exp(+\mathrm{i}x\mathsf{k}) \int_{\mathsf{x}} \mathrm{e}^{-\mathrm{i}\mathsf{k}\mathsf{x}}\Psi(\mathsf{x})\right]$$

The presence of the Gaussian ensures this integrand is Lebesgue integrable; the Dominated Convergence Theorem therefore applies, and the limit can be extracted out to the left. That allows the $\int_{\mathsf{x}}$ integral sign to move to the left; then by Fubini (or Tonelli?), the order of integrations can be swapped to do the $\int_{\mathsf{k}}$ integration first. Tidying a little gives:

$$\Psi(x) = \lim_{\epsilon\to 0} \int_{\mathsf{x}} \left[\int_{\mathsf{k}} \exp\left(-\frac{\epsilon}{2}\mathsf{k}^2 + \mathrm{i}x\mathsf{k}\right) \mathrm{e}^{-\mathrm{i}\mathsf{k}\mathsf{x}}\right] \Psi(\mathsf{x})$$

Completing the square:

$$\Psi(x) = \lim_{\epsilon\to 0} \int_{\mathsf{x}} \left[\int_{\mathsf{k}} \exp\left(-\frac{\epsilon}{2}\left[\mathsf{k} - \frac{\mathrm{i}x}{\epsilon}\right]^2 - \left[\frac{\mathrm{i}x}{\epsilon}\right]^2\right) \mathrm{e}^{-\mathrm{i}\mathsf{k}\mathsf{x}}\right] \Psi(\mathsf{x})$$

$$\Psi(x) = \lim_{\epsilon\to 0} \mathrm{e}^{-x^2/2\epsilon} \int_{\mathsf{x}} \left[\int_{\mathsf{k}} \exp\left(-\frac{\epsilon}{2}\left[\mathsf{k} - \frac{\mathrm{i}x}{\epsilon}\right]^2\right) \mathrm{e}^{-\mathrm{i}\mathsf{k}\mathsf{x}}\right] \Psi(\mathsf{x})$$

By Theorem 13.22, the translation: $\mathsf{k} - \frac{\mathrm{i}x}{\epsilon} \to \mathsf{k}$ yields the exponential shift $\mathrm{e}^{-\mathrm{i}(-\mathrm{i}x/\epsilon)\mathsf{x}} = \mathrm{e}^{-x\mathsf{x}/\epsilon}$:

$$\Psi(x) = \lim_{\epsilon\to 0} \mathrm{e}^{-x^2/2\epsilon} \int_{\mathsf{x}} \left[\mathrm{e}^{-x\mathsf{x}/\epsilon} \int_{\mathsf{k}} \exp\left(-\frac{\epsilon}{2}\mathsf{k}^2\right) \mathrm{e}^{-\mathrm{i}\mathsf{k}\mathsf{x}}\right] \Psi(\mathsf{x})$$

Evaluating the inner $\int_{\mathsf{k}}$ integral, using Lemma 13.25 :

$$\Psi\left(x\right)=\lim_{\epsilon\to0}\mathrm{e}^{-x^2/2\epsilon}\int_{\mathsf{x}}\left[\mathrm{e}^{-x\mathsf{x}/\epsilon}\frac{1}{\sqrt{\epsilon}}\exp\left(-\frac{1}{2\epsilon}\mathsf{x}^2\right)\right]\Psi\left(\mathsf{x}\right)$$

Collecting the exponential terms:

$$\Psi\left(x\right)=\lim_{\epsilon\to0}\int_{\mathsf{x}}\frac{1}{\sqrt{\epsilon}}\exp\left(-\frac{1}{2\epsilon}\mathsf{x}^2-\frac{x\mathsf{x}}{\epsilon}-\frac{x^2}{2\epsilon}\right)\Psi\left(\mathsf{x}\right)$$
$$\Psi\left(x\right)=\lim_{\epsilon\to0}\int_{\mathsf{x}}\frac{1}{\sqrt{\epsilon}}\exp\left[-\frac{1}{2\epsilon}\left(\mathsf{x}-x\right)^2\right]\Psi\left(\mathsf{x}\right)$$

Change of variable: $\mathsf{z}=\frac{1}{\sqrt{\epsilon}}\left(\mathsf{x}-x\right)\Rightarrow \mathrm{d}\mathsf{x}=\sqrt{\epsilon}\mathrm{d}\mathsf{z}$; $\Psi\left(\mathsf{x}\right)=\Psi\left(\sqrt{\epsilon}\mathsf{z}+x\right)$. But there is an important subtlety here, easy to miss. Because z is a dummy variable over the integral, it must be rational or real. And so, since x is now the difference between two rational or real numbers, it must also, without freedom, also rational or real. I indicate this by changing from the free variable notation x to the san serif x', thus giving:

$$\Psi\left(\mathsf{x}'\right)=\lim_{\epsilon\to0}\int_{\mathsf{z}}\sqrt{\epsilon}\frac{1}{\sqrt{\epsilon}}\exp\left[-\frac{1}{2}\mathsf{z}^2\right]\Psi\left(\sqrt{\epsilon}\mathsf{z}+\mathsf{x}'\right)$$

Finally, moving the limit back inside, and evaluating the standard integral:

$$=\int_{\mathsf{z}}\exp\left[-\frac{1}{2}\mathsf{z}^2\right]\lim_{\epsilon\to0}\Psi\left(\sqrt{\epsilon}\mathsf{z}+\mathsf{x}'\right)$$

$$\Psi\left(\mathsf{x}'\right)=2\pi\Psi\left(\mathsf{x}'\right)$$

And that result is indication of the normalising pre-factors, needed by the Fourier transform and its inverse, which I never considered throughout the chapter.

□

13.5 Supporting Theorems

Theorem 6. Translation Identity

$$\int_{\mathrm{d}x} f(x-a)\,\mathrm{e}^{-ipx} = \mathrm{e}^{-ipa}\int_{\mathrm{d}x} f(x)\,\mathrm{e}^{-ipx} \tag{13.22}$$

Proof. Change of variable: set $u = x - a$; and hence: $\mathrm{d}u = \mathrm{d}x$. Thus:

$$\int_{\mathrm{d}u} f(u)\,\mathrm{e}^{-ipu} = \int_{\mathrm{d}x} f(x-a)\,\mathrm{e}^{-ip(x-a)}$$

But:

$$\int_{\mathrm{d}u} f(u)\,\mathrm{e}^{-ipu} = \int_{\mathrm{d}x} f(x)\,\mathrm{e}^{-ipx}$$

Hence:

$$\begin{aligned}\int_{\mathrm{d}x} f(x)\,\mathrm{e}^{-ipx} &= \int_{\mathrm{d}x} f(x-a)\,\mathrm{e}^{-ip(x-a)}\\ &= \int_{\mathrm{d}x} f(x-a)\,\mathrm{e}^{-ipx}\mathrm{e}^{+ipa}\\ &= \mathrm{e}^{+ipa}\int_{\mathrm{d}x} f(x-a)\,\mathrm{e}^{-ipx}\\ \Rightarrow \mathrm{e}^{-ipa}\int_{\mathrm{d}x} f(x)\,\mathrm{e}^{-ipx} &= \int_{\mathrm{d}x} f(x-a)\,\mathrm{e}^{-ipx}\end{aligned}$$

□

Theorem 7. Scaling Identity

$$\int f(\lambda x)\,\mathrm{e}^{-ipx}\mathrm{d}x = \frac{1}{\lambda}\int f(x)\,\mathrm{e}^{-i\frac{p}{\lambda}x}\mathrm{d}x$$

Proof. Change variables: set $p\lambda = \varrho$; $u = \lambda x$; and hence: $\mathrm{d}u = \lambda\mathrm{d}x$. Thus:

$$\int f(u)\,\mathrm{e}^{-ipu}\mathrm{d}u = \int f(\lambda x)\,\mathrm{e}^{-ip\lambda x}\lambda\mathrm{d}x$$

But:

$$\begin{aligned}\int f(u)\,\mathrm{e}^{-ipu}\mathrm{d}u &= \int f(x)\,\mathrm{e}^{-ipx}\mathrm{d}x\\ \Rightarrow \int f(x)\,\mathrm{e}^{-ipx}\mathrm{d}x &= \int f(\lambda x)\,\mathrm{e}^{-ip\lambda x}\lambda\mathrm{d}x\\ \Rightarrow \int f(x)\,\mathrm{e}^{-ipx}\mathrm{d}x &= \int f(\lambda x)\,\mathrm{e}^{-ip\lambda x}\lambda\mathrm{d}x\end{aligned}$$

$$\Rightarrow \frac{1}{\lambda}\int f(x)\,\mathrm{e}^{-ipx}\mathrm{d}x = \int f(\lambda x)\,\mathrm{e}^{-ip\lambda x}\mathrm{d}x$$
$$\Rightarrow \frac{1}{\lambda}\int f(x)\,\mathrm{e}^{-i\frac{\varrho}{\lambda}x}\mathrm{d}x = \int f(\lambda x)\,\mathrm{e}^{-i\varrho x}\mathrm{d}x$$

□

Theorem 8. Fourier of Momentum Derivative

$$\mathcal{F}[-\mathrm{i}\partial f](p) = p\mathcal{F}[f](p) \tag{13.23}$$

Proof. For $(\partial f) \in S(\mathbb{R})$ (Schwartz space)

$$\begin{aligned}\mathcal{F}[-\mathrm{i}\partial f](p) &= \frac{1}{(2\pi)^{1/2}}\int_{\mathrm{d}x} \mathrm{e}^{-\mathrm{i}px}(-\mathrm{i}\partial f)(x)\\ &= \frac{1}{(2\pi)^{1/2}}\int_{\mathrm{d}x} \mathrm{e}^{-\mathrm{i}px}(-\mathrm{i}\partial f)(x)\end{aligned}$$

After integrating by parts, the surface term $\left[\mathrm{e}^{-\mathrm{i}px}f\right]_a^b$ goes away; because very rapidly decaying function, thus:

$$\begin{aligned}\mathcal{F}[-\mathrm{i}\partial f](p) &= \frac{1}{(2\pi)^{1/2}}\int_{\mathrm{d}x}(-\mathrm{i}p)(-\mathrm{i})\,\mathrm{e}^{-\mathrm{i}px}f(x) + \left[\mathrm{e}^{-\mathrm{i}px}f\right]_a^b\\ &= \frac{1}{(2\pi)^{1/2}}p\int_{\mathrm{d}x}\mathrm{e}^{-\mathrm{i}px}f(x)\\ &= p\mathcal{F}[f](p)\end{aligned}$$

□

Theorem 9. Fourier of Position Derivative

$$\mathcal{F}[xf](p) = +\mathrm{i}\partial\mathcal{F}[f](p) \tag{13.24}$$

Proof. For $(\partial f) \in S(\mathbb{R})$ (Schwartz space)

$$\begin{aligned}\mathcal{F}[xf](p) &= \frac{1}{(2\pi)^{1/2}}\int_{\mathrm{d}x}\mathrm{e}^{-\mathrm{i}px}xf(x)\\ &= \frac{1}{(2\pi)^{1/2}}\int_{\mathrm{d}x}\partial_p\left(\mathrm{i}\mathrm{e}^{-\mathrm{i}px}\right)xf(x)\\ &= \mathrm{i}\partial_p\frac{1}{(2\pi)^{1/2}}\int_{\mathrm{d}x}\left(\mathrm{e}^{-\mathrm{i}px}\right)xf(x)\\ &= \mathrm{i}\partial_p\mathcal{F}[xf](p)\end{aligned}$$

□

Lemma 1. Little Lemma

$$\mathcal{F}\left(\exp\left[-\frac{z}{2}x^2\right]\right)(y) = \frac{1}{\sqrt{z}}\exp\left[-\frac{1}{2z}y^2\right] \tag{13.25}$$

$x \in \mathbb{R}; \; z \in \mathbb{C} : Re\,(z) > 0$

Proof. Define $G_z\,(x) = \exp\left[-\frac{z}{2}x^2\right]$; G controlled by z, implies the derivative:

$$\partial_x G_z\,(x) = -zx \exp\left(-\frac{z}{2}x^2\right) = -zxG_z\,(x)$$

Now, for free variable p, apply the Fourier transform to both sides,

$$\mathcal{F}\left[\partial_x G_z\,(x)\right](p) = \mathcal{F}\left[-zxG_z\,(x)\right](p)$$

By Theorems 8 & 9, (13.23) applied on the left, and (13.24) on the right:

$$\mathrm{i}p\mathcal{F}\left[G_z\,(x)\right](p) = -\mathrm{i}z\partial_x\left(\mathcal{F}\left[G_z\,(x)\right]\right)(p)$$

This is an ordinary differential equation which can be solved by separation. Indefinite integral:

$$\begin{aligned}
\int \frac{\partial_x \mathcal{F}\left[G_z\,(x)\right]}{\mathcal{F}\left[G_z\,(x)\right]} &= \int -\frac{p}{z}\mathrm{d}p \\
\Rightarrow \ln\left|\mathcal{F}\left[G_z\right]\right| &= -\frac{p^2}{2z} + C \qquad\qquad : C \in \mathbb{C} \\
\Rightarrow \mathcal{F}\left[G_z\right] &= A\exp\left[-\frac{p^2}{2z}\right] \qquad : A \in \mathbb{C} \\
\Rightarrow \frac{1}{(2\pi)^{1/2}} \int_{\mathrm{d}x} \mathrm{e}^{-\mathrm{i}px} \exp\left[-\frac{z}{2}x^2\right] &= A\exp\left[-\frac{p^2}{2z}\right]
\end{aligned}$$

To find A set $p = 0$, $\mathrm{e}^{-\mathrm{i}px} = 1$

$$\begin{aligned}
\frac{1}{(2\pi)^{1/2}} \int_{\mathrm{d}x} \exp\left[-\frac{z}{2}x^2\right] &= A \times 1 \\
\frac{1}{(2\pi)^{1/2}} \sqrt{\frac{2\pi}{z}} &= A \\
\frac{1}{\sqrt{z}} &= A
\end{aligned}$$

This $\frac{1}{z}$ is true for $z \in \mathbb{R}$, but need for $z \in \mathbb{C}$, and so $\sqrt{z}$ is not simple. We have several possible branches. Since integral is holomorphic in z, differentiability is a stronger condition on the complex plane than on $\mathbb{R}$. Because holomorphic we can extend the result to complex z with real part > 0.

□